Dirac Mass Matrices in Gauge Field Theory of Horizontal Symmetry

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We investigate Dirac mass matrices derived in the gauge field theory of a horizontal symmetry generated by a central extension of the Pauli algebra. Through numerical analyses of the observed data of the charged fermion masses and the flavor mixing matrix of quarks, values of free parameters in the mass matrices are determined and several empirical relations are found among the Yukawa coupling constants. As one specific feature of the theory, we find different orderings in squared mass eigenvalues for the up and down quark sectors.

§1. Introduction

The Standard Model (SM) of particle physics possesses no principle to lay any restriction on the pattern of the Yukawa interactions. Nine complex coupling constants are treated as free parameters in every four sectors consisting of three generations of quarks and leptons. One way to describe order and variety of the generational structure is to postulate a gauge symmetry called the *horizontal symmetry*.[1),2)] In the previous paper,[3)] one of the present authors has proposed a gauge field theory of a new horizontal symmetry. The purpose of this paper is to investigate the Dirac mass matrices derived in the theory and make numerical analyses of the mass spectra of the charged fermions and the flavor mixing matrix (FMM) of quarks.

The horizontal (H) symmetry of the theory[3)] is postulated to be described by the Lie group generated by a central extension of the Pauli algebra which was found in investigating the FMM of quarks and leptons.[4),5)] The algebra consisting of four generators has the central element identified with the democratic matrix[6)] which can create hierarchical mass spectra of fundamental fermions. The number of the Yukawa coupling constants of the theory is reduced to 4/9 of that of the SM. Through spontaneous breakdown of the symmetry, the theory leads to the Dirac mass matrices $\mathcal{M}_f$ for the sector f with definite electric charge.

The matrix $\mathcal{M}_f$ with unique non-Hermitian structure possesses four unknown complex parameters. To deduce information on the f sector, we must solve the eigenvalue problem of the Hermitian matrix $\mathcal{M}_f\mathcal{M}_f^\dagger$. It should be noticed that unspecified quantities involved in these Hermitian matrices are reduced down to ten real parameters. Consequently, numerical analysis on the experimental data[7)] on the six masses and four FMM parameters of the quark sector enables us to determine the values of ten real parameters, from which we find several empirical relations among the Yukawa coupling constants.

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The Dirac mass matrices deduced from the Lagrangian density of the Yukawa interactions are shown explicitly in §2. We describe formalisms for the eigenvalue problem of $\mathcal{M}_f\mathcal{M}_f^\dagger$ in §3 and §4. Different mass orderings in the up and down quark sectors are explained in §5. In §6, values of the parameters in the mass matrices are determined by numerical analysis and specific empirical relations are found among the Yukawa coupling constants. Discussion is given in §7 and functional dependence among parameters and mass eigenvalues is examined in Appendix.

§2. Dirac mass matrices

In the low energy region less than the electroweak scale $v = 246\,\mathrm{GeV}$,[8] the gauge field theory of the H symmetry provides the effective theory for flavor phenomenology. Breakdowns of the H and electroweak symmetries lead to the Lagrangian density for Dirac masses of the quarks and leptons in the following forms[3]

$$\mathcal{L}_{\mathcal{M}}^{\mathrm{Y}} = \sum_{f=u,d} \bar{\Psi}_L^f \mathcal{M}_f \Psi_R^f + \sum_{f=\nu,e} \bar{\Psi}_L^f \mathcal{M}_f \Psi_R^f + \mathrm{h.c.} \tag{2·1}$$

in which $\Psi_{L,R}^f$ are chiral fermion fields and $\mathcal{M}_f$ is the Dirac mass matrix. For the up sectors ($f = u,\ \nu$) of the electroweak symmetry, the mass matrix is given by

$$\mathcal{M}_f = a_f\, I + \frac{1}{\sqrt{3}} b_{f1} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{3} b_{f2} \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{pmatrix} + c_f\, \breve{D} \tag{2·2}$$

where

$$\breve{D} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{2·3}$$

is the democratic element.[6] The four coefficients in the mass matrix are expressed in terms of the Yukawa coupling constants Y_{fi} and the vacuum expectation value v of the scalar field as $a_f = Y_{f1}\, v$, $b_{f1} = -Y_{f2}\, v$, $b_{f2} = Y_{f3}\, v$ and $c_f = 3Y_{f4}\, v$. For the down sectors ($f = d,\ e$), we find

$$\mathcal{M}_f = a_f\, I + b_{f1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{3}} b_{f2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + c_f\, \breve{D} \tag{2·4}$$

where $a_f = Y_{f1}\, v$, $b_{f1} = Y_{f2}\, v$, $b_{f2} = Y_{f3}\, v$ and $c_f = 3Y_{f4}\, v$.

In order to diagonalize the non-Hermitian mass matrix $\mathcal{M}_f$, it is necessary to have recourse to the bi-unitary transformation[9]

$$V_L^{f\dagger} \mathcal{M}_f V_R^f = \mathcal{M}_{f\mathrm{diagonal}}. \tag{2·5}$$

To derive the mass eigenvalues and the diagonalizing matrix V_L^f, it is necessary to solve the eigenvalue problem for the self-adjoint matrix $\mathcal{M}_f\mathcal{M}_f^\dagger$ as

$$\mathcal{M}_f\mathcal{M}_f^\dagger\, |\boldsymbol{v}^{(f)i}\rangle = m_i^{(f)2}\, |\boldsymbol{v}^{(f)i}\rangle \tag{2·6}$$

for each charged fermion sector ($f = u, d, e$). The diagonalizing matrix V_L^f is obtained in terms of the eigenvectors $|\boldsymbol{v}^{(f)i}\rangle$ and the FMM of the quark sector is constructed by

$$V = V_L^{u\dagger} V_L^d = \left(\langle \boldsymbol{v}^{(u)i} | \boldsymbol{v}^{(d)j} \rangle \right), \tag{2·7}$$

provided that the eigenvectors are arranged in increasing orders of the masses for up and down sectors.

§3. Eigenvalue problem 1: Quark FMM

To solve the eigenvalue problem for $\mathcal{M}_f \mathcal{M}_f^\dagger$, it turns out convenient to use the basis vectors

$$|\,1\,\rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad |\,2\,\rangle = \tfrac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad |\,3\,\rangle = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{3·1}$$

which are eigenvectors of the democratic element $\breve{D}$. With these bases, the eigenvector in (2·6) is expanded as

$$|\,\boldsymbol{v}^{(f)}\,\rangle = x_f |\,1\,\rangle + y_f |\,2\,\rangle + z_f |\,3\,\rangle. \tag{3·2}$$

For the up quark sector, (2·6) is rewritten for the coefficients of $|\boldsymbol{v}^{(u)}\rangle$ as

$$\begin{pmatrix} |a_u|^2 + 2|b_{u1}|^2 & 0 & \sqrt{2} C_u^* b_{u1} \\ 0 & |a_u|^2 & -\sqrt{2} a_u b_{u2}^* \\ \sqrt{2} C_u b_{u1}^* & -\sqrt{2} a_u^* b_{u2} & |C_u|^2 + 2|b_{u2}|^2 \end{pmatrix} \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} = m^{(u)2} \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \tag{3·3}$$

where $C_u = c_u + a_u$. Similarly, for the down quark sector, we obtain

$$\begin{pmatrix} |a_d|^2 & 0 & \sqrt{2} a_d b_{d2}^* \\ 0 & |a_d|^2 + 2|b_{d1}|^2 & -\sqrt{2} C_d^* b_{d1} \\ \sqrt{2} a_d^* b_{d2} & -\sqrt{2} C_d b_{d1}^* & |C_d|^2 + 2|b_{d2}|^2 \end{pmatrix} \begin{pmatrix} x_d \\ y_d \\ z_d \end{pmatrix} = m^{(d)2} \begin{pmatrix} x_d \\ y_d \\ z_d \end{pmatrix} \tag{3·4}$$

for the coefficients of $|\boldsymbol{v}^{(d)}\rangle$, where $C_d = c_d + a_d + b_{d1}$.

To clarify the counting of independent parameters in these equations and the FMM, we define phase factors by

$$C_u^* b_{u1} = |C_u b_{u1}| e^{i\mu_u},\ a_u b_{u2}^* = |a_u b_{u2}| e^{i\nu_u},\ a_d b_{d2}^* = |a_d b_{d2}| e^{i\mu_d},\ C_d^* b_{d1} = |C_d b_{d1}| e^{i\nu_d} \tag{3·5}$$

and introduce the diagonal phase matrix

$$P^f = \mathrm{diag}(\,e^{i\mu_f},\ e^{i\nu_f},\ 1\,) \tag{3·6}$$

for the f sector to adjust phase factors. By separating the diagonal phase matrices, the eigenvectors of (3·3) and (3·4) are found, respectively, in the forms

$$P^u \boldsymbol{u}_j^u = P^u N_j^u \begin{pmatrix} \sqrt{2}|C_u b_{u1}|(m_j^{(u)2} - |a_u|^2) \\ -\sqrt{2}|a_u b_{u2}|(m_j^{(u)2} - |a_u|^2 - 2|b_{u1}|^2) \\ (m_j^{(u)2} - |a_u|^2)(m_j^{(u)2} - |a_u|^2 - 2|b_{u1}|^2) \end{pmatrix} \tag{3·7}$$

and

$$P^d \boldsymbol{u}_j^d = P^d N_j^d \begin{pmatrix} \sqrt{2}|a_d b_{d2}|(m_j^{(d)2} - |a_d|^2 - 2|b_{d1}|^2) \\ -\sqrt{2}|C_d b_{d1}|(m_j^{(d)2} - |a_d|^2) \\ (m_j^{(d)2} - |a_d|^2)(m_j^{(d)2} - |a_d|^2 - 2|b_{d1}|^2) \end{pmatrix} \tag{3·8}$$

where $m_j^{(f)2}$ are eigenvalues of squared masses and N_j^f are the normalization constants. Then, with the orthogonal matrices

$$O_L^f = \left(\boldsymbol{u}_1^f,\ \boldsymbol{u}_2^f,\ \boldsymbol{u}_3^f\right) \tag{3·9}$$

consisting of the vectors $\boldsymbol{u}_j^f$, the FMM for the quark sector is calculated to be

$$V = O_L^{u\dagger} P O_L^d \tag{3·10}$$

with the diagonal phase matrix

$$P = \mathrm{diag}(\,e^{i\mu},\ e^{i\nu},\ 1\,) = P^{u\dagger} P^d, \quad \mu = \mu_d - \mu_u, \quad \nu = \nu_d - \nu_u. \tag{3·11}$$

This FMM includes unknown parameters of eight real numbers and two phases. The secular equations for the eigenvalue problems in (3·3) and (3·4) work to fix six real parameters in terms of the mass eigenvalues. Consequently, two real numbers and two phases remain unspecified in the FMM for the quark sector.

§4. Eigenvalue problem 2: Mass spectra of charged fermions

Since the secular equations for both of the eigenvalue problems in (3·3) and (3·4) take the same form, the suffix f is omitted for all quantities in this section and the Appendix. In terms of the shifted variable $s = m^2 - |a|^2$, the secular equation is obtained as

$$s^3 - (|C|^2 - |a|^2 + 2|b|^2)\,s^2 - 2(|ab|^2 - 2|b_1 b_2|^2)\,s + 4|ab_1 b_2|^2 = 0 \tag{4·1}$$

where

$$|b|^2 = |b_1|^2 + |b_2|^2. \tag{4·2}$$

Let us solve this equation by the Cardano method. Introducing the dimensionless quantities

$$P = \frac{1}{3}\frac{|C|^2 - |a|^2 + 2|b|^2}{|ab_1 b_2|^{\frac{2}{3}}}, \quad Q = \frac{2}{3}\frac{|a|^2|b|^2 - 2|b_1 b_2|^2}{|ab_1 b_2|^{\frac{4}{3}}}, \tag{4·3}$$

and changing the variable by $s = |ab_1b_2|^{\frac{2}{3}}\,(t+P)$, we obtain the reduced proper equation without the second order term as follows:

$$t^3 - 3(P^2+Q)t - 2P^3 - 3PQ + 4 = 0. \tag{4·4}$$

One solution of this equation is determined as $t = t_+ + t_-$ by the sum of two quantities t_+ and t_- which are subject to the relations

$$t_\pm^3 = \frac{1}{2}\left(2P^3 + 3PQ - 4 \pm i\sqrt{|D|}\right) \tag{4·5}$$

where

$$D = -16P^3 - 3P^2Q^2 - 24PQ - 4Q^3 + 16. \tag{4·6}$$

In the analysis below, it is appropriate to use the polar representation $t_+ = \rho \mathrm{e}^{i|\theta|}$ and $t_- = \rho \mathrm{e}^{-i|\theta|}$ in which ρ and θ are expressed, in terms of P and Q, as

$$\rho = \sqrt{P^2+Q}, \quad \tan 3|\theta| = \frac{\sqrt{|D|}}{2P^3 + 3PQ - 4}. \tag{4·7}$$

Then, the three solutions of (4·4) are derived to be

$$\begin{aligned} t_1 &= \omega t_+ + \omega^2 t_- &&= 2\rho\cos(|\theta| + \tfrac{2\pi}{3}) \\ t_2 &= \omega^2 t_+ + \omega t_- &&= 2\rho\cos(|\theta| + \tfrac{4\pi}{3}) \\ t_3 &= t_+ + t_- &&= 2\rho\cos\theta \end{aligned} \tag{4·8}$$

where $\omega = \exp(i2\pi/3)$.

In this way, we have solved the eigenvalue problems in (3·3) and (3·4) obtaining the squared masses as follows:

$$\begin{aligned} m_1^2 &= |a|^2 + \tfrac{1}{3}\left(|C|^2 - |a|^2 + 2|b|^2\right)\left[1 + 2\sqrt{1+\delta}\cos(|\theta| + \tfrac{2\pi}{3})\right], \\ m_2^2 &= |a|^2 + \tfrac{1}{3}\left(|C|^2 - |a|^2 + 2|b|^2\right)\left[1 + 2\sqrt{1+\delta}\cos(|\theta| + \tfrac{4\pi}{3})\right], \\ m_3^2 &= |a|^2 + \tfrac{1}{3}\left(|C|^2 - |a|^2 + 2|b|^2\right)\left[1 + 2\sqrt{1+\delta}\cos\theta\right], \end{aligned} \tag{4·9}$$

where the parameter

$$\delta = 6\frac{|a|^2|b|^2 - 2|b_1b_2|^2}{[|C|^2 - |a|^2 + 2|b|^2]^2} \tag{4·10}$$

is introduced to simplify the expressions.

As confirmed below, the magnitude of the angle $|\theta|$ must be sufficiently small for the mass spectra to have hierarchical structure. Here, it is crucially important to note that the squared masses have orderings $m_1^2 < m_2^2 < m_3^2$ and $m_2^2 < m_1^2 < m_3^2$, respectively, for $\theta > 0$ and $\theta < 0$.

The squared masses in (4·9) depend on the four real quantities $|a|^2$, $|b_1|^2$, $|b_2|^2$ and $|C|^2$. In the present analysis, we interpret inversely that $|b_1|^2$, $|b_2|^2$ and $|C|^2$ are functions of the masses and the parameter $|a|^2$. Then, all quantities for the quark

FMM are determined in terms of the observed quark masses and the independent parameter $|a|$. (See the Appendix).

In the following sections, we make numerical analysis of the quark FMM by using experimental values of quark masses as inputs and by adjusting the independent parameters $|a_u|$ and $|a_d|$.

§5. Mass orderings of the up and down quark sectors

For numerical analyses below, we use the quark masses at the energy scale of the Z boson, i.e., $m_Z = 91.2\,\mathrm{GeV}$. The values calculated by the renormalization group equations[10] are given as follows:[11]

$$\begin{aligned} &m_u = 1.27^{+0.50}_{-0.42}\,\mathrm{MeV}, \quad &&m_c = 0.619{\pm}0.084\,\mathrm{GeV}, \quad &&m_t = 171.7{\pm}3.0\,\mathrm{GeV}, \\ &m_d = 2.90^{+1.24}_{-1.19}\,\mathrm{MeV}, \quad &&m_s = 55^{+16}_{-15}\,\mathrm{MeV}, \quad &&m_b = 2.89{\pm}0.09\,\mathrm{GeV}. \end{aligned} \tag{5·1}$$

The observed FMM of the quark sector has the prominent feature that the matrix elements decrease rapidly for each step away from the diagonal. To reproduce such characteristics, both of the orthogonal matrices O_L^u and O_L^d in (3·9) must approximately be close to the unit matrix. Accordingly, as a step for the FMM analysis, it is reasonable to examine numerically which of the solutions characterized by $\theta > 0$ and $\theta < 0$ in (4·9) and the associated eigenvectors in (3·7) and (3·8) can bring the orthogonal matrix nearer to the unit matrix.

Let us apply the positive-θ solution with the experimental mass values in (5·1) to examine the orthogonal matrices. For the down quark sector, it is proved that all of the diagonal elements of the matrix O_L^d can approach to the unit for small value of $|a_d|$. Contrastingly, for the up quark sector, some of the diagonal elements of the matrix O_L^u is shown to be quite smaller than 1 for any value of $|a_u|$. The situation reverses completely for the negative-θ solution. Numerical calculations with the negative-θ solution show that, while O_L^u approaches to the unit matrix by adjusting $|a_u|$, O_L^d can not be made close to the unit matrix for any value of $|a_d|$.

Accordingly, it is necessary to choose the positive- and negative-θ solutions, respectively, for the down and up quark sectors to reproduce the experimental results of the quark FMM. The masses and state vectors of the observed down quark members (d, s, b) must be described by the positive-θ solution with normal ordering as follows:

$$\begin{aligned} m_d = m_1^{(d)}, \qquad m_s = m_2^{(d)}, \qquad m_b = m_3^{(d)}; \\ |\boldsymbol{v}_d\rangle = |\boldsymbol{v}^{(d)1}\rangle, \quad |\boldsymbol{v}_s\rangle = |\boldsymbol{v}^{(d)2}\rangle, \quad |\boldsymbol{v}_b\rangle = |\boldsymbol{v}^{(d)3}\rangle. \end{aligned} \tag{5·2}$$

As for the observed up quark members (u, c, t), their masses and state vectors have to be identified with the quantities of the negative-θ solutions with partly-reversed ordering as follows:

$$\begin{aligned} m_u = m_2^{(u)}, \qquad m_c = m_1^{(u)}, \qquad m_t = m_3^{(u)}; \\ |\boldsymbol{v}_u\rangle = |\boldsymbol{v}^{(u)2}\rangle, \quad |\boldsymbol{v}_c\rangle = |\boldsymbol{v}^{(u)1}\rangle, \quad |\boldsymbol{v}_t\rangle = |\boldsymbol{v}^{(u)3}\rangle. \end{aligned} \tag{5·3}$$

In the present theory, it is crucial to accept these interpretations of the solutions of the eigenvalue problems in (2·6).

§6. Hierarchical structure of the Yukawa coupling constants

As shown in the Appendix, the quantities $|b_{f1}|$, $|b_{f2}|$ and $|C_f|$ are expressed in terms of the masses and the adjustable parameter $|a_f|^2$ in the hierarchical approximation ($m_3^{(f)2} \gg m_1^{(f)2}, m_2^{(f)2}$). Using these results and accepting the interpretations in (5·2) and (5·3), we make numerical analysis of the quark FMM. The best fit to the FMM data of the Particle Data Group[7] is obtained with the following values of the four parameters as

$$|a_u| = 30.4, \quad |a_d| = 13.2, \quad \mu = 0.96, \quad \nu = 2.32 \tag{6·1}$$

which lead to the magnitude of the elements of the quark FMM as

$$|V| = \begin{pmatrix} 0.974210 & 0.225705 & 0.003595 \\ 0.225473 & 0.973343 & 0.041511 \\ 0.008723 & 0.040746 & 0.999132 \end{pmatrix}, \tag{6·2}$$

and the Jarlskog invariant measure[12]

$$J = 3.1 \times 10^{-5} \tag{6·3}$$

for the CP violation. The values of all parameters for the best fit found above are listed in Table I.

Table I. Values of ten parameters

up quark (MeV)	down quark (MeV)	phases
$|a_u| = 30.4$	$|a_d| = 13.2$	$\mu = 0.96$
$|b_{u1}| = 831$	$|b_{d1}| = 92.1$	$\nu = 2.32$
$|b_{u2}| = 63800$	$|b_{d2}| = 818$	
$|C_u| = 146000$	$|C_d| = 2650$	

Results in Table I show clearly that the real parameters satisfy the hierarchical orderings $|a_f|^2 \ll |b_{f1}|^2 \ll |b_{f2}|^2 \ll |C_f|^2$ for each quark sector. By making more careful comparison among them, we find approximate relations

$$\frac{|b_{u1}|}{|b_{d2}|} \sim 1, \quad \frac{|b_{d2}|}{|b_{d1}|} \sim 9^1, \quad \frac{|b_{u2}|}{|b_{u1}|} \sim 9^2 \tag{6·4}$$

which result readily in interesting empirical relations

$$\frac{|Y_{u2}|}{|Y_{d3}|} \sim 1, \quad \frac{|Y_{d3}|}{|Y_{d2}|} \sim 9^1, \quad \frac{|Y_{u3}|}{|Y_{u2}|} \sim 9^2, \tag{6·5}$$

among the four Yukawa coupling constants.

The largest quantity in Table I is $|C_f|$ for both quark sectors. Note that the original parameter c_f in the mass matrix $\mathcal{M}_f$ can appear only through the quantity C_f, as $C_u = c_u + a_u$ and $C_d = c_d + a_d + b_{d1}$, in the Hermitian matrix $\mathcal{M}_f \mathcal{M}_f^\dagger$. Due to

this feature which works to decrease the number of unknown quantities in $\mathcal{M}_f\mathcal{M}_f^\dagger$, we can do nothing but determine $|c_f|$ approximately $|c_u| \approx |C_u|$ and $|c_d| \approx |C_d|$.

Using $v = 246\,\mathrm{GeV}$ and the data in Table I, we are able to fix the magnitudes of the Yukawa coupling constants of the quark sectors as in Table II. In the somehow crude approximation for $|c_f|$, the constant $|Y_{f4}|$ have comparatively large uncertainty.

Table II. Yukawa coupling constants

f	u	d	e
$\|Y_{f1}\|$	1.2×10^{-4}	5.4×10^{-5}	3.5×10^{-5}
$\|Y_{f2}\|$	3.4×10^{-3}	3.7×10^{-4}	4.1×10^{-4}
$\|Y_{f3}\|$	2.6×10^{-1}	3.3×10^{-3}	3.6×10^{-3}
$\|Y_{f4}\|$	2.0×10^{-1}	3.6×10^{-3}	1.6×10^{-3}

Thus far, the Yukawa coupling constants of quark sectors are calculated so as to recreate the observed data of the quark FMM. This approach is not applicable, as it stands, to the lepton sector. Here, let us find an approximate scheme with less numbers of adjustable parameters based on the benefit of hindsight on the empirical relations in (6·5) and apply it to analyze the masses of charged leptons.

For its purpose, we introduce a new variable β by the equations

$$|b_{d1}| = 9\beta, \quad |b_{d2}| = |b_{u1}| = 9^2\beta, \quad |b_{u2}| = 9^4\beta, \tag{6·6}$$

which verify all of the relations in (6·4), and reinvestigate the quark masses and FMM in terms of seven quantities $|a_u|, |a_d|, |C_u|, |C_d|, \mu, \nu$ and β. It turns out possible to

Table III. Values of parameters for approximate estimation

up quark (MeV)	down quark (MeV)	common parameters
$\|a_u\| = 30$	$\|a_d\| = 13$	$\mu = 0.95,\ \ \nu = 2.3$
$\|C_u\| = 142000$	$\|C_d\| = 2620$	$\beta = 10$

explain all observed data within the range of experimental errors. In fact, by using the values of seven quantities in Table III, we obtain

$$\begin{aligned} m_u &= 1.20\,\mathrm{MeV}, \qquad m_c = 0.628\,\mathrm{GeV}, \qquad m_t = 169.6\,\mathrm{GeV}, \\ m_d &= 2.87\,\mathrm{MeV}, \qquad m_s = 53.8\,\mathrm{MeV}, \qquad m_b = 2.86\,\mathrm{GeV}, \end{aligned} \tag{6·7}$$

for the six quark masses and

$$|V| = \begin{pmatrix} 0.973984 & 0.226589 & 0.003509 \\ 0.226452 & 0.973161 & 0.040962 \\ 0.008616 & 0.040199 & 0.999155 \end{pmatrix}, \quad J = 2.9 \times 10^{-5} \tag{6·8}$$

for the magnitudes of the quark FMM elements and the Jarlskog parameter.

For both sectors of the down quark and the charged lepton, the Dirac mass matrices take the common forms in (2·4). To analyze the masses of the charged leptons in this approximate scheme, we assume that the empirical equations related to the down quarks in (6·4) hold also in the charged lepton sector. Then, following the equations in (6·6), we introduce a new variable β_e by the relations

$$|b_{e1}| = 9\beta_e, \quad |b_{e2}| = 9^2\beta_e \tag{6·9}$$

and analyze the charged lepton masses in terms of the three parameters $|a_e|$, $|C_e|$ and β_e. Numerical estimation shows readily that the charged lepton masses[10),11)]

$$m_e = 0.4866\,\mathrm{MeV}, \quad m_\mu = 102.7\,\mathrm{MeV}, \quad m_\tau = 1746\,\mathrm{MeV} \tag{6·10}$$

can be reproduced by adjusting the parameters as follows:

$$|a_e| = 8.537, \quad |C_e| = 1197, \quad \beta_e = 11.06 \tag{6·11}$$

from which the Yukawa coupling constants for the charged lepton sector can be fixed provided that $|c_e| \approx |C_e|$. The results are included in Table II. Fig.1 shows behavior of the Yukawa coupling constants of charged fermion sectors.

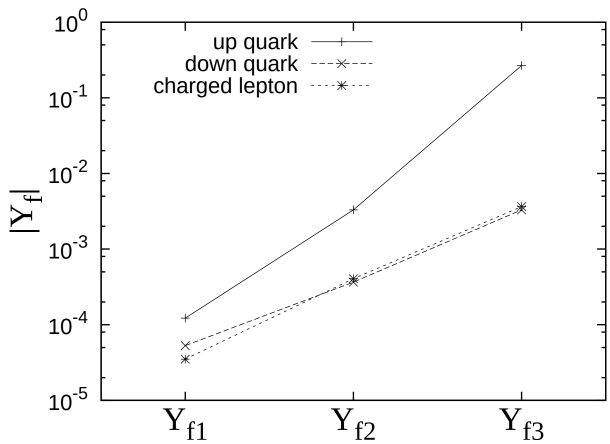

Fig. 1. The Yukawa coupling constants $|Y_{f1}|$, $|Y_{f2}|$ and $|Y_{f3}|$.

§7. Discussion

We have investigated the non-Hermitian mass matrices $\mathcal{M}_f$ in (2·2) and (2·4) deduced in the gauge field theory of the horizontal symmetry generated by the central extension of the Pauli algebra. While the matrix $\mathcal{M}_f$ possesses four complex unknown parameters, the Hermitian combination $\mathcal{M}_f\mathcal{M}_f^\dagger$ depends on four real numbers and two phases. Owing to this fact, we are able to make numerical analyses on the problem of the six quark masses and the quark FMM with four parameters.

To solve the problem effectively, six real parameters are interpreted as functions of quark masses and remaining two parameters. Using the observed masses as inputs and adjusting values of two independent parameters and two phases, we have

reproduced the observed FMM and succeeded to calculate the Yukawa coupling constants. By careful examination of the numerical results, empirical relations in (6·5) are found among the coupling constants for the quark sector. It is beyond the scope of the present theory to elucidate physical implications of those relations.

The estimated values of the Yukawa coupling constants in Table II show novel orderings with sizable amount of variation. It should be noticed that the largest ratio between the coupling constants, $|Y_{u3}|/|Y_{u1}| \sim |Y_{u4}|/|Y_{u1}| \sim 10^3$, is considerably smaller than that between the observed quark masses $m_t/m_u \sim 10^5$. This means that varieties observed directly in the low energy flavor physics are much reduced at the level of the coupling constants of the Yukawa interactions.

All results of numerical analyses so far were obtained in the hierarchical approximation ($m_3^{(f)2} \gg m_1^{(f)2}, m_2^{(f)2}$) by using the formulae in the Appendix which express $|b_{f1}|, |b_{f2}|$ and $|C_f|$ as functions of $|a_f|^2$. It should be noticed, however, that we are able to find almost the same results by adjusting the values of real quantities $|b_{f1}|$, $|b_{f2}|$, $|C_f|$ and $|a_f|^2$ and two phases μ and ν.

The squared masses of the charged fermions are derived in the exact formula in (4·9). Depending on the sign of the angle θ, the formula has the solutions with normal and partly-reversed orderings in (5·2) and (5·3). As confirmed in §5, we have to describe the down and up quark states by the normal and partly-reversed solutions, respectively, in order to reproduce the observed FMM. This is one of unique features of the present formalism for the horizontal gauge symmetry.

We made a speculative analysis of the charged lepton masses by postulating hypothetical relations (6·9) as analogues of the equations for the down quark sector in (6·6). To proceed a full investigation on the charged and neutral lepton sectors, however, it is required to solve the combined eigenvalue problem which involves not only the Dirac mass matrices but also the Majorana mass matrix deduced for the neutrino sector.[3)] We will study this problem in near future.

Appendix A
—— *Dependence of $|b_1|$, $|b_2|$ and $|C|$ on $|a|$* ——

In this scheme, we interpret the real quantities $|b_1|$, $|b_2|$ and $|C|$ as functions of the quark masses and the parameter $|a|$. For its purpose, it is appropriate to define a mass M for reference by the relation

$$m_3^2 + m_2^2 + m_1^2 = |C|^2 + 2|b|^2 + 2|a|^2 = 3M^2. \qquad \text{(A·1)}$$

Evidently observed data in (5·1) show hierarchical orderings of squared masses, M^2, $m_3^2 \gg m_2^2$, m_1^2, for each sector. To deduce convenient relations for numerical analyses, we express all quantities as power series of M^2.

In the hierarchical limit, P and $|C|^2$ take very large values. Equation (4·7) shows that $(\tan 3|\theta|)^2$ can be decomposed in the power series of M^{-2}, leading to the approximate relation

$$(3\theta)^2 \simeq 4\frac{|ab_1b_2|^2}{M^6} + \frac{1}{3}\frac{\left(|a|^2|b|^2 - 2|b_1b_2|^2\right)^2}{M^8}. \qquad \text{(A·2)}$$

Similarly, the quantity δ defined by (4·10) is approximated by

$$\delta \simeq \frac{2}{3}\frac{|a|^2|b|^2 - 2|b_1 b_2|^2}{M^4}. \tag{A·3}$$

Decomposition of the eigenvalues m_1^2 and m_2^2 in (4·9) with respect to M^{-2} results in the following expressions

$$m_2^2 + m_1^2 \simeq 2|a|^2 - M^2\delta, \quad m_2^2 - m_1^2 \simeq 2\sqrt{3}M^2\theta. \tag{A·4}$$

Eliminating θ and δ from these equations, we obtain the relations

$$|b_1 b_2|^2 = \frac{3}{4|a|^2}(m_2^2 - |a|^2)(|a|^2 - m_1^2)M^2, \quad |b|^2 = \frac{3}{2}\frac{|a|^4 - m_1^2 m_2^2}{|a|^4}M^2. \tag{A·5}$$

Then, substitution of the last expression for $|b|^2$ into the defining equation (A·1) allows to express $|C|^2$ in the form

$$|C|^2 = 3\frac{m_1^2 m_2^2}{|a|^4}M^2 - 2|a|^2. \tag{A·6}$$

Finally, to distinguish between $|b_1|$ and $|b_2|$, we introduce an extra quantity κ by

$$|b_1|^2 = \frac{1}{2}|b|^2 - M\kappa, \quad |b_2|^2 = \frac{1}{2}|b|^2 + M\kappa. \tag{A·7}$$

From (A·5), κ is calculated to be

$$\kappa = \sqrt{\frac{3}{4|a|^2}\left(\frac{3}{4}\frac{(|a|^4 - m_2^2 m_1^2)^2}{|a|^6}M^2 - (m_2^2 - |a|^2)(|a|^2 - m_1^2)\right)}. \tag{A·8}$$

Consequently, the quantities $|C|$, $|b_1|$ and $|b_2|$ are determined as the functions of quark masses and the parameter $|a|$ for each sector. The quark FMM is now expressible in terms of the observed mass values and the four adjustable parameters $|a_u|$, $|a_d|$, μ and ν.

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